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Quasiprobabilities Based on Squeezed States

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We introduce quasiprobabilities based on the so-called squeezed states to represent the density operator of an oscillator. Such representations become especially useful for oscillators designed to display, strong excitation notwithstanding, pronounced quantum features such as squeezing of the quantum fluctuations of certain observables below the limit characteristic of coherent states.

KEY WORDS: Quasiprobabilities; convolution identities; squeezed states; coherent states; subharmonic generation.

1. INTRODUCTION

Among the quantum states[1] an oscillator is capable of assuming, the well-known coherent states are the most nearly classical ones with respect to the statistics they assign to the position x and the momentum p. Neither x nor p is sharp in a coherent state, but their uncertainty product takes on the minimum value compatible with their commutator. Moreover, the respective variances equal one another if expressed in their natural quantum units. Pictorially speaking, the quantum fluctuations inherent in a coherent state may be represented by a circular disk around the phase locus of the state with the disk area determined by Planck's constant.

There is an abundance of experiments, most notably in quantum optics, in which a linear or nonlinear oscillator becomes sufficiently highly and coherently excited for its quantum behavior to have a close-to-classical character. It is typical, then, that the oscillator can be described by, if not a single, a mixture of coherent states with a weight corresponding to small

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position and momentum spreads. The weight in question, the so-called P function, then tends to be of narrow support but well behaved with respect to its "phase space" arguments. In such cases the P function behaves, in many respects, like an ordinary classical phase space density.[1,2]

There is, on the other hand, no lack of processes endowing oscillators with a high degree of excitation and yet very nonclassical characteristics. In particular, great effort is currently being devoted to squeezing the variance of either the position or the momentum below the coherent-state value at the expense, of course, of increasing the other of the two variances as demanded by the uncertainty principle (see refs. 3 and 4 for review).

An ideal squeezed state[5,6] resembles a coherent state inasmuch as it is associated with a point in classical phase space and has the minimal product of position and momentum uncertainties. It differs from a coherent state in that the phase space area of uncertainty around the locus is elliptic rather than circular in shape. The quantum fluctuations are enhanced along the major axis of the ellipse and reduced along the minor axis. The nonclassical nature of a squeezed state is reflected, for instance, in the fact that the corresponding density operator does not possess a diagonal representation with respect to coherent states with a well-behaved P function.

The strict realization of any minimum uncertainty state, coherent or squeezed, is not an easy matter for experimenters. However, as already mentioned, an oscillator excited to nearly classical behavior can usually be described by a mixture of coherent states with a well-behaved P function of narrow phase space support. The same will in general not be true for an oscillator squeezed to anisotropic variances, but in that case a representation by a mixture of squeezed states with a smooth weight function should be possible.

This paper is organized as follows. We briefly review coherent states in Section 2 and squeezed states in Section 3. In Section 4 we introduce a one-parameter family of squeezed-state-based quasiprobability densities generalizing a similar family based on coherent states.[7–9] In particular, the analogues of the familiar P, Q, and Wigner functions are members of that family. Section 5 is devoted to the explicit relation between coherent-state-based and squeezed-state-based quasiprobabilities. Every coherent-state-based quasiprobability can be obtained from every squeezed-state-based one (and vice versa) as a convolution of the latter with a suitable Gaussian. The Gaussian in question is determined by the squeezing parameter and the indices specifying the quasiprobabilities within the respective families. As a first aplication, we calculate in Section 6 the squeezed-state-based quasiprobabilities representing a single squeezed state. Section 7 provides an important tool for further applications by establishing the rules for constructing evolution equations for quasiprobabilities from a given evolution equation of the density operator. Finally, we illustrate our concepts for the problem of subthreshold subharmonic generation in Section 8.

2. COHERENT STATES

As is well known, the harmonic part H_0 of any oscillator Hamiltonian H can be diagonalized with the help of a pair of non-Hermitian operators a and a^+ which obey $[a, a^+]=1$ and give $H_0=\hbar\omega(a^+a+\frac{1}{2})$. The excitation quanta described by H_0 are annihilated by a and created by its adjoint a^+, as is obvious from their action on the n-quantum eigenstate $|n\rangle$ of a^+a or H_0, $a|n\rangle=n^{1/2}|n-1\rangle$.

Coherent states of an oscillator may be defined[1] as eigenstates of the annihilator a,

$$a|\alpha\rangle=\alpha|\alpha\rangle \tag{2.1}$$

with the dimensionless eigenvalue α an arbitrary complex number. It is intuitive to associate a point in classical phase space with the amplitude α through the expectation values of the position x and the momentum $p=m\dot{x}$,

$$\alpha=\tfrac{1}{2}(\langle x\rangle/x_0+i\langle p\rangle/p_0) \tag{2.2}$$

where $x_0=(\hbar/2m\omega)^{1/2}$ and $p_0=(\hbar m\omega/2)^{1/2}$ are natural quantum units of the position and the momentum. If expressed in these units, the coherent-state variances of x and p become equal,

$$\langle(\Delta x/x_0)^2\rangle=\langle(\Delta p/p_0)^2\rangle=1 \tag{2.3}$$

Both variances are independent of the amplitude α. Their product has the minimal value allowed by the uncertainty principle.

A useful equivalent definition[1] describes the coherent state $|\alpha\rangle$ as generated from the vacuum $|0\rangle$ by the unitary displacement operator

$$D(\alpha)=e^{\alpha a^+ - \alpha^* a} \tag{2.4}$$

$$|\alpha\rangle=D(\alpha)\,|0\rangle \tag{2.5}$$

The operator $D(\alpha)$ is quite appropriately called the displacement operator in view of the identity

$$D^+(\alpha)\,aD(\alpha)=a+\alpha \tag{2.6}$$

which, incidentally, yields an immediate proof of the equivalence of (2.5) and (2.1).

The coherent states are overcomplete and permit the following well-known resolution of the unit operator

$$1 = \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha| \tag{2.7}$$

where the integral extends over the whole complex α plane. Similarly, a large class of density operators allow for a diagonal representation of the form

$$\rho = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha| \tag{2.8}$$

with the so-called P function as a weight. If such a P function exists as a reasonably well-behaved function of α and α^* or at least as a tempered distribution,[2,10] its moments give the means of normally ordered products of creation and annihilation operators as

$$\langle a^{+m} a^n \rangle = \operatorname{tr} a^{+m} a^n \rho = \int d^2\alpha \, \alpha^{*m} \alpha^n P \tag{2.9}$$

The latter property suggests that we interpret P as a quasiprobability density.

A much better behaved quasiprobability which in fact exists for any density operator and is nonnegative throughout the α plane is the socalled Q function,

$$Q(\alpha) = \frac{1}{\pi} \langle\alpha| \, \rho \, |\alpha\rangle \tag{2.10}$$

The moments of Q are the means of antinormally ordered products $a^n a^{+m}$. A continuous class of quasiprobabilities can be defined through their convolution with a normalized Gaussian,[7-9]

$$Q(\alpha) = \frac{1}{\pi\varepsilon} \int d^2\beta \, W_\varepsilon(\beta) \exp\left(-\frac{1}{\varepsilon} |\alpha - \beta|^2 \right), \qquad 0 \leqslant \varepsilon \leqslant 1 \tag{2.11}$$

In the limit $\varepsilon = 0$, the Gaussian in (2.11) becomes a delta function such that $W_0(\alpha)$ is just $Q(\alpha)$. As other important special cases, (2.11) contains the Wigner function for $\varepsilon = \frac{1}{2}$ and the P function for $\varepsilon = 1$. In fact, the identity (2.11) for $\varepsilon = 1$ can be obtained directly by taking the expectation value with respect to a coherent state on both sides in (2.8) and recalling the scalar product of two coherent states

$$\langle\alpha | \beta\rangle = \exp(\alpha^* \beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2) \tag{2.12}$$

When the convolution in (2.11) is unfolded by a Fourier transform it becomes obvious that the quasiprobabilities $W_\varepsilon(\alpha)$ tend to become increasingly singular as the width ε of the Gaussian in (2.11) is increased toward unity.

Means of normally ordered products of a and a^+ can be calculated with the help of $W_\varepsilon(\alpha)$ through[7-9]

$$\langle a^{+m} a^n \rangle = \int d^2\alpha \left(\alpha^* + (1-\varepsilon) \frac{\partial}{\partial \alpha} \right)^m \left(\alpha + (1-\varepsilon) \frac{\partial}{\partial \alpha^*} \right)^n W_\varepsilon(\alpha) \quad (2.13)$$

Strange as the densities $W_\varepsilon(x)$ for general ε may appear, these functions have a certain usefulness with respect to the dynamics of linear and nonlinear oscillators. A given von Neumann or master equation of motion for the density operator $\rho(t)$ can be translated into a partial differential operation for $W_\varepsilon(\alpha, t)$, $\dot{W}_\varepsilon = L_\varepsilon W_\varepsilon$, where the generator L_ε is a differential operator with respect to α and α^* and contains the width ε as a parameter. Approximate or even exact solutions of that differential equation are often found to be most easily accessible when ε is confined to certain restricted subintervals of $0 \leqslant \varepsilon \leqslant 1$.[11]

3. SQUEEZED STATES

With the help of the unitary squeezing operator

$$S(\eta) = \exp[\tfrac{1}{2}\eta^* a^2 - \tfrac{1}{2}\eta a^{+2}) \quad (3.1)$$

we introduce squeezed states as[5,6]

$$|\alpha, \eta\rangle = D(\alpha)\, S(\eta)\, |0\rangle \quad (3.2)$$

allowing α and η to be arbitrary complex numbers. The states $|\alpha, \eta\rangle$ thus form a four-parameter family. Evidently, for $\eta = 0$ the states (3.2) reduce to ordinary coherent states, $|\alpha, 0\rangle = |\alpha\rangle$.

The physical meaning of the parameter η,

$$\eta = re^{i2\theta} \quad (3.3)$$

becomes obvious when we use the identity

$$S^+(\eta)\, aS(\eta) = a \cosh r - a^+ e^{i2\theta} \sinh r \quad (3.4)$$

in calculating expectation values of observables with respect to the states (3.2). We immediately infer

$$\langle \alpha, \eta |\, a\, | \alpha, \eta \rangle = \langle 0 |\, S^+(\eta)(a + \alpha)\, S(\eta)\, | 0 \rangle = \alpha \quad (3.5)$$

Similarly, the mean of any linear combination of a and a^+ (e.g., x or p) is independent of η and thus equal to the mean with respect to the coherent state $|\alpha\rangle$. In order to meet observable differences between the coherent state $|\alpha\rangle$ and the squeezed state $|\alpha, \eta\rangle$, we must therefore consider second- or higher-order moments of a and a^+. To that end, it is appropriate to introduce Hermitian linear combinations of a and a^+ which, in contrast to x and p, are designed such as to reveal the role of the phase θ distinguished by the squeezing parameter η,

$$\begin{aligned} X_\theta &= ae^{-i\theta} + a^+e^{i\theta} \\ X_{\theta+\pi/2} &= (1/i)(ae^{-i\theta} - a^+e^{i\theta}) \end{aligned} \tag{3.6}$$

These special observables obey, due to (3.4),

$$\begin{aligned} S^+(\eta)\, X_\theta S(\eta) &= e^{-r}X_\theta \\ S^+(\eta)\, X_{\theta+\pi/2} S(\eta) &= e^{+r}X_{\theta+\pi/2} \end{aligned} \tag{3.7}$$

They are the only nontrivial linear combinations which reproduce, to within a scale factor, under the unitary transformation $S(\eta)$, rather than getting mixed.

In pursuing our goal of clarifying the physical meaning of η, it is worthwhile to discuss, as a final preparation, the unitary transformation under $S(\eta)$ of the displacement $D(\alpha)$. From (3.4) we find the transformed displacement operator to be again a displacement operator,

$$S^+(\eta)\, D(\alpha)\, S(\eta) = D(\alpha(\eta)) \tag{3.8}$$

the new displacement being

$$\alpha(\eta) = \alpha \cosh r + \alpha^* e^{i2\theta} \sinh r \tag{3.9}$$

It follows that the squeezed state (3.2) can be rewritten as

$$|\alpha, \eta\rangle = S(\eta)\, D(\alpha(\eta))\, |0\rangle = S(\eta)\, |\alpha(\eta)\rangle \tag{3.10}$$

i.e., as the coherent state $|\alpha(\eta)\rangle$ acted upon by the unitary squeezer $S(\eta)$. Equipped with this reinterpretation of $|\alpha, \eta\rangle$, we consider the variances

$$\begin{aligned} \langle\alpha, \eta|\, (\Delta X_\theta)^2\, |\alpha, \eta\rangle &= \langle\alpha(\eta)|\, S^+(\Delta X_\theta)^2 S\, |\alpha(\eta)\rangle \\ &= e^{-2r}\langle\alpha(\eta)|\, (\Delta X_\theta)^2\, |\alpha(\eta)\rangle \\ &= e^{-2r} \end{aligned} \tag{3.11}$$

and, similarly,

$$\langle\alpha, \eta|\, (\Delta X_{\theta+\pi/2})^2\, |\alpha, \eta\rangle = e^{+2r} \tag{3.12}$$

It is now obvious that η is appropriately called the squeezing parameter. Its modulus r measures the degree of asymmetry between the variances of X_θ

and $X_{\theta+\pi/2}$. The angles θ and $\theta+\pi/2$, on the other hand, give the directions of maximum squeezing and maximum stretching, respectively. As for a coherent state, the product of the squeezed-state variances (3.11), (3.12) takes on the minimum value compatible with the uncertainty principle.

It may be worthwhile to point out that the squeezed state $|\alpha, \eta\rangle$ can be interpreted as a coherent state with respect to the reversely [with respect to (3.4)] squeezed annihilator

$$\begin{aligned} a(\eta) &= S(\eta)\, aS^{+}(\eta) \\ &= a \cosh r + a^{+} e^{i2\theta} \sinh r \end{aligned} \tag{3.13}$$

with the eigenvalue (3.9). Indeed, by using (3.10), we have

$$\begin{aligned} a(\eta)\, |\alpha, \eta\rangle &= S(\eta)\, aS^{+}(\eta)\, S(\eta)\, |\alpha(\eta)\rangle \\ &= \alpha(\eta)\, |\alpha, \eta\rangle \end{aligned} \tag{3.14}$$

4. QUASIPROBABILITIES BASED ON SQUEEZED STATES

For any fixed value of the squeezing parameter η the set of complex numbers α corresponds to an overcomplete set of nonorthogonal states $|\alpha, \eta\rangle$. In analogy with (2.7), we have as a resolution of unity[5,6]

$$1 = (1/\pi) \int d^2\alpha\, |\alpha; \eta\rangle\langle\alpha; \eta| \tag{4.1}$$

which is most easily proven by rewriting the squeezed state according to (3.10) and checking that the transformation $\alpha \to \alpha(\eta)$ given in (3.9) is canonical and thus has a Jacobian equal to unity. The integral in (4.1) can therefore be changed into one over $\alpha(\eta)$ at fixed η,

$$(1/\pi) \int d^2\alpha(\eta)\, S(\eta)\, |\alpha(\eta)\rangle\langle\alpha(\eta)|\, S^{+}(\eta)$$

which is indeed equal to the unit operator due to (2.7) and the unitary of $S(\eta)$.

As a natural generalization of (2.8), we now introduce the diagonal representation of a density operator as[2]

$$\begin{aligned} \rho &= \int d^2\alpha\, P(\alpha(\eta); \eta)\, |\alpha, \eta\rangle\langle\alpha, \eta| \\ &= \int d^2\alpha\, P(\alpha; \eta)\, S(\eta)\, |\alpha\rangle\langle\alpha|\, S^{+}(\eta) \end{aligned} \tag{4.2}$$

[2] Note that the integral over the complex α plane can as well be extended over the complex $\alpha(\eta)$ plane. It is for the sake of convenience that we define $P(\alpha(\eta); \eta)$ to have the natural arguments $\alpha(n)$ and $\alpha^*(\eta)$ rather than α and α^*

Like the ordinary P function $P(\alpha)=P(\alpha(0);0)$, the density $P(\alpha(\eta);\eta)$ need not necessarily exist for a given density operator. The generalization of the Q function (2.10).

$$Q(\alpha(\eta);\eta)=\frac{1}{\pi}\langle\alpha;\eta|\,\rho\,|\alpha;\eta\rangle \tag{4.3}$$

must always exist, however. By taking expectation values with respect to a squeezed state in (4.2) and using the scalar product

$$\begin{aligned}\langle\beta;\eta\,|\,\alpha;\eta\rangle &= \langle\beta(\eta)\,|\,\alpha(\eta)\rangle\\ &=\exp[\beta^*(\eta)\,\alpha(\eta)-\tfrac{1}{2}|\beta(\eta)|^2-\tfrac{1}{2}|\alpha(\eta)|^2]\end{aligned} \tag{4.4}$$

we find that $P(\alpha;\eta)$ and $Q(\alpha;\eta)$ are related to one another through the convolution with a Gaussian. Finally, by considering the width of the latter Gaussian as a free parameter, we generate, in parallel to (2.11), a whole class of new quasiprobabilities through

$$Q(\alpha(\eta);\eta)=\frac{1}{\pi\varepsilon}\int d^2\beta\, e^{-|\beta-\alpha(\eta)|^2/\varepsilon}\,W_\varepsilon(\beta;\eta),\qquad 0\leqslant\varepsilon\leqslant 1 \tag{4.5}$$

Obviously, $W_\varepsilon(\alpha;\eta)$ interpolates between $P(\alpha;\eta)=W_1(\alpha;\eta)$ and $Q(\alpha;\eta)=W_0(\alpha;\eta)$. We should also note that $W_\varepsilon(\alpha;\eta)$ bears a relation to $a(\eta)$, $a^+(\eta)$ analogous to that of $W_\varepsilon(\alpha)$ to a, a^+. In particular, the moments $\langle a^+(\eta)^m\,a(\eta)^n\rangle$ are given by (2.13) with $W_\varepsilon(\alpha;\eta)$ replacing $W_\varepsilon(\alpha)$.

5. SQUEEZED-STATE-BASED VERSUS COHERENT-STATE-BASED QUASIPROBABILITIES

The convolution identity (4.5) relates different quasiprobabilities within a family determined by a fixed value of the squeezing parameter η. We now propose to relate the quasiprobabilities with nonzero η to those pertaining to $\eta=0$, i.e., to coherent states. To that end it is convenient to represent the density operator in normally ordered from,[1,2,12]

$$\rho=\int\frac{d^2\xi}{\pi}\,e^{-\xi a^+}\,e^{\xi^* a}\,\chi(\xi) \tag{5.1}$$

where the characteristic function

$$\chi(\xi)=\operatorname{tr} e^{-\xi^* a}e^{\xi a^+}\rho \tag{5.2}$$

is the two-dimensional Fourier transform of the Q function (2.10). Indeed, by taking expectation values with respect to a coherent state in (5.1), we obtain

$$Q(\alpha)=\frac{1}{\pi^2}\int d^2\xi\, e^{-\xi\alpha^*+\xi^*\alpha}\,\chi(\xi) \tag{5.3}$$

In complete analogy to (5.1), we may represent the density operator with the operators $a(\eta)$ and $a^+(\eta)$, defined in (3.13), arranged in normal order,

$$\rho=\int\frac{d^2\xi}{\pi}\, e^{-\xi a^+(\eta)}\, e^{\xi^* a(\eta)}\,\chi(\xi;\eta) \tag{5.4}$$

Since the squeezed states $|\alpha,\eta\rangle$ are coherent with respect to the squeezed annihilator $a(\eta)$, the characteristic function

$$\chi(\xi;\eta)=\operatorname{tr} e^{-\xi^* a(\eta)}e^{\xi a^+(\eta)}\rho \tag{5.5}$$

is nothing but the Fourier transform of the squeezed-state Q function $Q(\alpha(\eta);\eta)$ defined in (4.3).

In order to relate the two characteristic functions $\chi(\xi)$ and $\chi(\xi;\eta)$, we insert the inverse of the relation (3.13),

$$a=a(\eta)\cosh r-a^+(\eta)e^{i2\theta}\sinh r$$

into (5.5) and use the Baker–Campbell–Hausdorff formula[3] to establish antinormal order with respect to $a(\eta)$ and $a^{+(\eta)}$. As a first step, we unite the two exponentials in (5.2) into a single one,

$$\chi(\xi)=\exp(-\tfrac{1}{2}\xi\xi^*)\operatorname{tr}\rho\exp[-a(\eta)\,\xi^*(\eta)+a^+(\eta)\,\xi(\eta)] \tag{5.6}$$

with $\xi(\eta)$ as in (3.9). Splitting again to the new antinormal order, we obtain the desired relation

$$\chi(\xi)=\{\exp[-\tfrac{1}{2}\xi\xi^*+\tfrac{1}{2}\xi(\eta)\,\xi^*(\eta)]\}\,\chi(\xi(\eta);\eta) \tag{5.7}$$

Interestingly, the Gaussian on the right-hand side in (5.7) is not bounded throughout the complex ξ plane. Nonetheless, the Fourier transforms of both characteristic functions, being Q functions, exist and we therefore have

$$\begin{aligned}Q(\alpha)=\int\frac{d^2\xi}{\pi}\int\frac{d^2\beta}{\pi}&\exp[-\xi\alpha^*+\xi^*\alpha+\xi(\eta)\beta^*-\xi^*(\eta)\beta]\\&\times\exp[-\tfrac{1}{2}\xi\xi^*+\tfrac{1}{2}\xi(\eta)\,\xi(\eta)^*]\,Q(\beta;\eta)\end{aligned} \tag{5.8}$$

[3] For any pair of operators c and d, the commutator of which commutes both with c and d, we have $e^c\cdot e^d=e^{c+d+[c,d]/2}$.

We should note that the order of the integrations in (5.8) cannot be reversed, due to the nonboundedness of the Gaussian just mentioned.

It will be useful to generalize the relation just obtained by substituting (2.11) and (4.5) for $Q(\alpha)$ and $Q(\alpha; \eta)$, respectively, so as to relate the coherent-state-based density $W_\varepsilon(\alpha)$ to the squeezed-state-based density $W_\varepsilon(\alpha; \eta)$. By simply repeating the argument leading to (5.8), (a similar relation was obtained in ref. 9)

$$W_\varepsilon(\alpha) = \int \frac{d^2\xi}{\pi} \frac{d^2\beta}{\pi} \exp[-\xi(\alpha^* - \beta^*) + \xi^*(\alpha - \beta)]$$
$$\times \{\exp[-(\tfrac{1}{2} - \varepsilon)\, \xi\xi^* + (\tfrac{1}{2} - \varepsilon')\, \xi(\eta)\, \xi^*(\eta)]\}\, W_{\varepsilon'}(\beta(\eta); \eta)$$
$$0 \leqslant \varepsilon,\ \varepsilon' \leqslant 1 \qquad (5.9)$$

This relation should prove a convenient starting point for applications of squeezed-state-based quasiprobabilities. It obviously contains (5.8) as the special case $\varepsilon = \varepsilon' = 0$.

Evidently, by decreasing the width parameter ε and/or increasing ε', we counteract the tendency of the Gaussian in (5.9) to blow up as $\xi \to \infty$. It is in fact easy to check that the Gaussian remains bounded if

$$(\varepsilon' - \tfrac{1}{2})e^{-2r} \geqslant \varepsilon - \tfrac{1}{2} \qquad \text{for} \quad \varepsilon' \geqslant \tfrac{1}{2} \qquad (5.10a)$$

or

$$(\tfrac{1}{2} - \varepsilon')e^{2r} \leqslant \tfrac{1}{2} - \varepsilon \qquad \text{for} \quad \varepsilon' \leqslant \tfrac{1}{2} \qquad (5.10b)$$

With ε and ε' chosen to obey (5.10), the ξ integral in (5.9) can be carried out. To given a compact appearance to the resulting relation between $W_\varepsilon(\alpha)$ and $W_\varepsilon(\alpha; \eta)$, we introduce a Hermitian two by two matrix A characterizing the Gaussian in question as

$$\tfrac{1}{2}(\xi^*, \xi) A \begin{pmatrix} \xi \\ \xi^* \end{pmatrix} = (\tfrac{1}{2} - \varepsilon)\, \xi\xi^* - (\tfrac{1}{2} - \varepsilon')\, \xi(\eta)\, \xi^*(\eta) \qquad (5.11)$$

We can then write the simple convolution identity

$$W_\varepsilon(\alpha) = \int \frac{d^2\beta}{\pi} \left(\frac{1}{\det A}\right)^{1/2} \exp\left\{\tfrac{1}{2}(-\alpha^* + \beta^*, \alpha - \beta) A^{-1} \begin{pmatrix} \alpha - \beta \\ -\alpha^* + \beta^* \end{pmatrix}\right\}$$
$$\times W_{\varepsilon'}(\beta(\eta); \eta) \qquad (5.12)$$

This expression loses its meaning, though, when ε and ε' do not obey (5.10), since $\det A < 0$ then. Incidentally, (5.12) contains (2.11) as the special case $\eta = 0$, $\varepsilon = 0$.

For further reference, we note the explicit form of the matrix A,

$$\begin{aligned} A_{11} &= A_{22} = \varepsilon' \cosh 2r - \varepsilon - \sinh^2 r \\ A_{12} &= A_{21}^* = (\varepsilon' - \tfrac{1}{2}) e^{i2\theta} \sinh 2r \end{aligned} \tag{5.13}$$

6. QUASIPROBABILITIES REPRESENTING A PURE SQUEEZED STATE

The simplest example of a density operator for which the squeezed-state-based quasiprobabilities $W_\varepsilon(\alpha(\eta), \eta)$ but not the coherent-state-based ones $W_\varepsilon(\alpha)$ exist for all values of the width parameter ε in $0 \leqslant \varepsilon \leqslant 1$ is the projector on a squeezed state,

$$\rho = |\alpha_0, \eta\rangle\langle\alpha_0, \eta| \tag{6.1}$$

Due to (4.2), the P function pertaining to the particular value of the squeezing parameter distinguished in (6.1) reads

$$P(\alpha(\eta); \eta) = \delta^2(\alpha(\eta) - \alpha_0(\eta)) = \delta^2(\alpha - \alpha_0) \tag{6.2}$$

Similarly easily accessible is the corresponding Q function. With the help of (4.3), (4.4), we have

$$Q(\alpha(\eta); \eta) = \frac{1}{\pi} \exp[-|\alpha(\eta) - \alpha_0(\eta)|^2] \tag{6.3}$$

Finally, (4.5) yields

$$W_\varepsilon(\alpha(\eta); \eta) = \frac{1}{\pi(1-\varepsilon)} \exp\left(-\frac{|\alpha(\eta) - \alpha_0(\eta)|^2}{1-\varepsilon}\right) \tag{6.4}$$

and this includes, as it must, the P and Q functions as the special cases $\varepsilon = 1$ and $\varepsilon = 0$, respectively. Note that the densities $W_\varepsilon(\alpha(\eta), \eta)$ exist for all values of ε in $0 \leqslant \varepsilon \leqslant 1$.

The coherent-state-based quasiprobabilities $W_\varepsilon(\alpha)$ can now be calculated with the help of (5.9), most easily by setting $\varepsilon' = 1$ and using (6.2). The result is

$$W_\varepsilon(\alpha) = \frac{1}{\pi} \left(\frac{1}{\det A}\right)^{1/2} \exp\left\{\frac{1}{2}(-\alpha^* + \alpha_0, \alpha - \alpha_0) A^{-1} \begin{pmatrix} \alpha - \alpha_0 \\ -\alpha^* + \alpha_0 \end{pmatrix}\right\} \tag{6.5}$$

with the matrix A defined in (5.11) taken at $\varepsilon' = 1$. The range of ε for which the density $W_\varepsilon(\alpha)$ exists can be inferred from (5.10) with $\varepsilon' = 1$ as

$$\varepsilon \leqslant \tfrac{1}{2}(1 + e^{-2r}) \tag{6.6}$$

It follows, in particular, that the coherent-state-based P function will not exist if $r = |\eta| > 0$.

7. RELATIONS BETWEEN EVOLUTION EQUATIONS

We now proceed to the dynamics of our oscillator, assuming an equation of motion for the density operator of the form

$$\dot{\rho}(t) = L\rho(t) \tag{7.1}$$

In all cases of practical interest the generator L acts linearly on $\rho(t)$ and is representable by a fourfold series as

$$L_\rho = \sum_{mnpq} L_{mnpq} a^m a^{+n} \rho a^p a^{+q} \tag{7.2}$$

with c-number coefficients such that $L^*_{mnpq} = L_{pqmn}$.

If the oscillator moves reversibly, the generator L is given in terms of the Hamiltonian as $L\rho = -i[H, \rho]/\hbar$. Otherwise, L contains additional pieces accounting for irreversible influences.

The operator equation (7.1) implies a c-number differential equation for each of the quasiprobabilities $W(\alpha(\eta), t; \eta)$,

$$\begin{aligned} \partial_t W_\varepsilon(\alpha, t; \eta) &= l^{(\varepsilon,\eta)} W_\varepsilon(\alpha, t; \eta) \\ l^{(\varepsilon,\eta)} &= \sum_{mnpq} l^{(\varepsilon,\eta)}_{mnpq} \left(\frac{\partial}{\partial\alpha}\right)^m \left(\frac{\partial}{\partial\alpha^*}\right)^n \alpha^{*p} \alpha^q \end{aligned} \tag{7.3}$$

the coefficients $l^{(\varepsilon,\eta)}_{mnpq}$ being uniquely determined by the L_{mnpq} in (7.2). As is well known, the differential operator (7.3) can be obtained most easily for the coherent-state-based P and Q functions, i.e., for $\eta = 0$ and $\varepsilon = 1$ or $\varepsilon = 0$, by assuming the density operator ρ antinormally or normally ordered with respect a^+ and a.[12] For instance, by looking at the antinormally ordered form of ρ and using the commutator $[a, a^+] = 1$, we easily verify the translation rule

$$\begin{aligned} a\rho &\to \alpha P, & \rho a^+ &\to \alpha^* P \\ a^+\rho &\to (\alpha^* - \partial/\partial\alpha) P, & \rho a &\to (\alpha - \partial/\partial\alpha^*) P \end{aligned} \tag{7.4}$$

from which the c-number generator $l^{(1,0)}$ can be built up. Having once found $l^{(1,0)}$, we may establish the generators $l^{(\varepsilon,0)}$ pertaining to all the other coherent-state-based densities $W_\varepsilon(\alpha)$ by the further translation[7,8,9,11]

$$\begin{aligned} \alpha &\to \alpha + (1-\varepsilon)\, \partial/\partial\alpha^* \\ \partial/\partial\alpha &\to \partial/\partial\alpha \end{aligned} \tag{7.5}$$

as is immediately checked with the help of the convolution identity (2.11).

Alternatively, we may wish to switch from the coherent-state-based P function to the squeezed-state-based density $W_\varepsilon(\alpha(\eta); \eta)$. The rule for translating, say, $l^{(1,0)}$, into $l^{(\varepsilon,\eta)}$ can be worked out from (5.12). To find the differential operation on $W_\varepsilon(\alpha; \eta)$ corresponding to multiplication of $P(\alpha)$ with α, we represent $\alpha P(\alpha)$ by (5.12) and note, as a first step, the identity

$$\alpha G(\alpha-\beta) = (\beta + A_{11}\, \partial/\partial\beta^* - A_{12}\, \partial/\partial\beta)\, G(\alpha-\beta) \tag{7.6}$$

obeyed by the Gaussian kernel in (5.12). By partially integrating, we shift the differentiations from the Gaussian onto $W_\varepsilon(\beta(\eta); \eta)$ and finally use the linear transformation $\beta \to \beta(\eta)$ to express the resulting differential operator in terms of the natural independent variables of $W_\varepsilon(\beta(\eta); \eta)$. In this way we obtain

$$\begin{aligned} \alpha P(\alpha) &\to \Big\{\alpha \cosh r - \alpha^* e^{i2\theta} \sinh r \\ &\qquad + \varepsilon e^{i2\theta} \sinh r \frac{\partial}{\partial\alpha} + (1-\varepsilon)\cosh r \frac{\partial}{\partial\alpha^*}\Big\}\, W_\varepsilon(\alpha; \eta) \\ \frac{\partial}{\partial\alpha} P(\alpha) &\to \left(\cosh r \frac{\partial}{\partial\alpha} + e^{-i2\theta} \sinh r \frac{\partial}{\partial\alpha^*}\right) W_\varepsilon(\alpha; \eta) \end{aligned} \tag{7.7}$$

Of course, by combining (7.4) and (7.7), it is possible to directly translate $a\rho$, $a^+\rho$, etc., into differential operations on $W_\varepsilon(\alpha; \eta)$.

An interesting and useful corollary of the first of the rules (7.7) arises for the expectation values of normally ordered products of a and a^+, i.e., the moments of the coherent-state-based P function,

$$\langle a^{+m} a^n \rangle = \int d^2\alpha\, \{\cdot\}^{*m} \{\cdot\}^n\, W_\varepsilon(\alpha; \eta) \tag{7.8}$$

with the curly brackets from (7.7). Note that the order of $\{\cdot\}$ and $\{\cdot\}^*$ in (7.8) is immaterial since the differential operator $\{\cdot\}$ commutes with its complex conjugate. The result (7.8) reduces, as it must, to (2.13) in the case of no squeezing, $\eta = 0$.

8. SUBHARMONIC GENERATION

We here propose to illustrate the usefulness of squeezed-state-based quasiprobabilities for a simple and well-known process, subthreshold subharmonic generation.[13,14] The system to be dealt with is a damped oscillator subjected to an undepletable monochromatic field, the latter coupling to the squared amplitude of the oscillation. To stay clear of unnecessary complications, we employ the so-called rotating wave approximation and adopt the rotating frame in which the external field appears to be stationary. We then have to solve the following master equation for the density operator:

$$\begin{aligned} \dot{\rho} &= -(i/\hbar)[H, \rho] + \Lambda\rho \\ H &= \hbar\delta a^+ a + i\hbar f(a^2 - a^{+2})/2 \\ \Lambda\rho &= \gamma\{[a\rho, a^+] + [a, \rho a^+]\} \end{aligned} \tag{8.1}$$

The three parameters δ, f, and γ are the detuning between the frequencies of the free oscillator and the external field, the amplitude of the external field, and a damping constant, respectively; they are all frequencies in dimension. In more realistic models the undepletable field f is replaced by a dynamical pump mode.[13,14]

By using (7.4), (7.7) for $\varepsilon = 1$ we can translate the master equation (8.1) into a differential equation of motion for the squeezed-state-based P function $P(\alpha, \eta)$ with $\eta = re^{i2\theta}$ a free parameter,

$$\begin{aligned} \dot{P} &= lP \\ l &= \frac{\partial}{\partial\alpha}[(i\Omega + \gamma)\alpha + F\alpha^*] \\ &\quad + \gamma \sinh^2 r \frac{\partial^2}{\partial\alpha\,\partial\alpha^*} + (\gamma e^{i2\theta}\cosh r \sinh r - \tfrac{1}{2}F)\frac{\partial^2}{\partial\alpha^2} \\ &\quad + \text{c.c.} \\ \Omega &= \delta \cosh 2r + f \sinh 2r \sin 2\theta \\ F &= -i\delta \sinh 2r\, e^{i2\theta} + f(\cosh^2 r - \sinh^2 r\, e^{i4\theta}) \end{aligned} \tag{8.2}$$

Formally, the partial differential equations (8.2) looks like the Fokker–Planck equation of a classical Gaussian Markov process; it actually is a genuine Fokker–Planck equation only if the coefficients of the second-order derivatives form a nonnegative matrix, the so-called diffusion matrix.

It is easy to check and quite important to realize that the diffusion matrix in (8.2) has one positive and one negative eigenvalue if we set $\eta = 0$, i.e., try to work with the coherent-state-based P function. We may visualize that situation as a tendency of $P(\alpha)$ to shrink in width with respect to a particular direction in the α plane and to broaden diffusively with respect to the orthogonal direction. In other words, the model under study tends to produce squeezing.

With one diffusion coefficient negative, the coherent-state-based P function is not an appropriate tool for describing the dynamics of our model. Even if well behaved initially, $P(\alpha, t)$ is doomed to perish once its width in any direction has shrunk to zero. We are thus led back to the conclusion that no mixture of coherent states can represent a squeezed state.

There is no reason, on the other hand, to doubt the possibility of representing a density operator as a mixture of suitable squeezed states when it contains the effects of squeezing. We must expect and do indeed find the diffusion matrix in (8.2) strictly positive for appropriately chosen values of the squeezing parameter η. Figure 1 depicts the region in the complex η plane for which this is the case. To every point of that region there corresponds a squeezed-state-based P function which, if it exists initially, is guaranteed existence at all subsequent times.

The remaining freedom in choosing η can be exploited so as to given to the Fokker–Planck equation (8.2) a simpler structure. We might, for instance, determine η such that either the diffusion matrix or the drift matrix becomes isotropic in the α plane. Alternatively, we might enforce potential conditions for drift and diffusion. We have found it most convenient, though, to require the stationary solution of (8.2) to be isotropic, i.e., to be a function of the product $\alpha\alpha^*$ only. That condition determines the direction of maximal squeezing through the angle θ as

$$e^{i2\theta} = (\gamma - i\delta)/(\gamma^2 + \delta^2)^{1/2} \tag{8.3}$$

and the magnitude of squeezing by

$$\sinh^2 r = \frac{(\gamma^2 + \delta^2)^{1/2} - (\gamma^2 + \delta^2 - f^2)^{1/2}}{2(\gamma^2 + \delta^2 - f^2)^{1/2}} \tag{8.4}$$

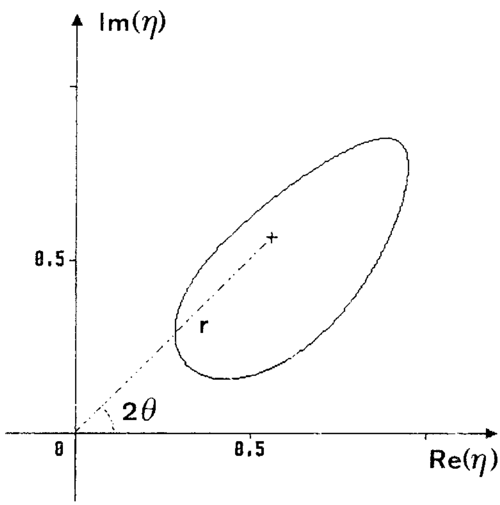

Fig. 1. Within the egg-shaped region of the complex η plane the diffusion matrix in (8.2) is strictly positive. The cross distinguishes the point (8.3), (8.4) corresponding to the isotropic stationary P function. The plot pertains to $\delta/\gamma = -1$, $f/\gamma = 1.3$.

The stationary P function then takes the simple Gaussian form

$$P(\alpha; \eta) = (\pi \sinh^2 r)^{-1} \exp(-\alpha\alpha^*/\sinh^2 r) \tag{8.5}$$

With the help of (7.8), we can now find all stationary expectation values. In particular, the mean oscillation amplitude turns out to vanish,

$$\langle a \rangle = \langle a^+ \rangle = 0 \tag{8.6}$$

while the second-order moments read

$$\begin{aligned} \langle a^2 \rangle &= -\frac{f}{2(\gamma + i\delta)} (1 + 2\langle a^+ a \rangle) \\ \langle a^+ a \rangle &= f^2/2(\gamma^2 + \delta^2 - f^2) \end{aligned} \tag{8.7}$$

The breakdown of our linear model at the threshold $f = (\gamma^2 + \delta^2)^{1/2}$ is manifest in (8.4) and (8.7). Near and above that threshold, pump depletion must be taken into account.[13,14] We shall not discuss above-threshold subharmonic generation here, since the usefulness of squeezed-state-based quasiprobabilities is nicely illustrated in the context of the linear model already.

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